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Some global well-posedness results for the compressible barotropic viscous fluid flow

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1 Introduction

This article is a brief survey of the paper [3], which is a joint work with Yoshihiro Shibata, Professor of Waseda University.

We consider the compressible Navier-Stokes equations:

$$\begin{cases} \rho_t + \operatorname{div}((\rho + \rho_*)\mathbf{u}) = 0 & \text{in } \Omega \times (0, T), \\ (\rho + \rho_*)(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) - \mu \Delta \mathbf{u} - \nu \nabla \operatorname{div} \mathbf{u} + \nabla \mathbf{p} = 0 & \text{in } \Omega \times (0, T), \\ \mathbf{u} = 0 & \text{on } \Gamma \times (0, T), \\ (\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0) & \text{in } \Omega \end{cases} \quad (1.1)$$

where Ω is an exterior domain of the 3-dimensional Euclidean space \mathbb{R}^3 , that is $\Omega = \mathbb{R}^3 \setminus \mathcal{O}$ with some bounded domain \mathcal{O} in \mathbb{R}^3 and Γ the boundary of Ω which is assume to be a smooth compact hyper-surface. Here, ρ_* is a positive constant describing the mass density of the reference body, and μ and ν are positive constants describing the first and second viscosity coefficients, respectively. Moreover, $\rho = \rho(x, t)$ with $x = (x_1, x_2, x_3) \in \Omega$, is a unknown function, $\rho_* + \rho$ being the density field, $\mathbf{u} = (u_1(x, t), u_2(x, t), u_3(x, t))$ the unknown velocity field, \mathbf{p} the unknown pressure field, (ρ_0, \mathbf{u}_0) prescribed initial data, and $\mathbf{u}|_\Gamma = 0$ is non-slip boundary condition. We consider only the barotropic case, that is

$$\mathbf{p} = P(\rho_* + \rho),$$

where $P(s)$ is a C^∞ function defined for $s > 0$ satisfying the condition:

$$\rho_1 \leq P'(\rho + \rho_*) \leq \rho_2 \quad \text{for } |\rho| \leq \rho_*/2$$

with some positive constants ρ_1 and ρ_2 .

The purpose of this article is to prove the global well-posedness of (1.1). Moreover, it is proved the optimal decay property.

When $\Omega = \mathbb{R}^3$ or Ω is a 3-dimensional exterior domain, Matsumura and Nishida [6, 7] proved the global well-posedness of problem (1.1) under the assumption that the H^3 norm of initial data ρ_0 and \mathbf{u}_0 are small enough. And also, Matsumura and Nishida [6, 7] and later on Deckelnick [1, 2] proved some convergence rate of solutions to the stationary solutions by using the energy method. The optimal decay properties were obtained by

Ponce [8] when $\Omega = \mathbb{R}^N$ and by Kobayashi and Shibata [5] when Ω is a 3-dimensional exterior domain with the help of the L_p - L_q decay estimate for the linearized equations.

In 2002, Kawashita [4] proved the global well-posedness of (1.1) when $\Omega = \mathbb{R}^N$ ($N \geq 2$) under the assumption that the H^s norm of initial data with $s = [N/2] + 1$ are small enough, in particular, $s = 2$ when $N = 2$ and 3. Later, Wang and Tan [10] proved the optimal decay estimate of Kawashita's solution when $\Omega = \mathbb{R}^3$ with the help of the L_p - L_q decay estimate due to Ponce [8]. The purpose of this paper is to prove the same results for the initial boundary value problem in 3-dimensional exterior domains as that for the Cauchy problem obtained by Kawashita [4] and Wang and Tan [10].

The following theorem shows the global well-posedness of (1.1).

Theorem 1.1. *Let Ω be a 3-dimensional exterior domain whose boundary Γ is a C^2 compact hyper-surface. Then, there exists a small positive number δ such that if initial data $(\rho_0, \mathbf{u}_0) \in H^2(\Omega)^4$ satisfy the smallness assumption: $\|(\rho_0, \mathbf{u}_0)\|_{H^2(\Omega)} \leq \delta$ and the compatibility condition: $\mathbf{u}_0|_\Gamma = 0$, then problem (1.1) with $T = \infty$ admits unique solutions ρ and \mathbf{u} with*

$$\begin{aligned} \rho &\in C^0((0, \infty), H^2(\Omega)) \cap C^1([0, \infty), H^1(\Omega)), \\ \rho_t &\in L_2((0, \infty), H^1(\Omega)), \quad \nabla \rho \in L_2((0, \infty), H^1(\Omega)^3) \\ \mathbf{u} &\in C^0([0, \infty), H^2(\Omega)^3) \cap C^1([0, \infty), L_2(\Omega)^3), \\ \mathbf{u}_t &\in L_2((0, \infty), H^1(\Omega)^3), \quad \nabla \mathbf{u} \in L_2((0, \infty), H^2(\Omega)^9) \end{aligned} \quad (1.2)$$

possessing the estimate:

$$\begin{aligned} &\sup_{0 < s < t} \|(\rho, \mathbf{u})(\cdot, s)\|^2 + \sup_{0 < s < t} \|(\nabla \rho, \rho_s, \nabla \mathbf{u})(\cdot, s)\|^2 + \sup_{0 < s < t} \|(\nabla^2 \rho, \nabla \rho_s, \nabla^2 \mathbf{u})(\cdot, s)\|^2 \\ &\quad + \int_0^t (\|\nabla \mathbf{u}(\cdot, s)\|^2 + \|\rho_s(\cdot, s)\|^2 + \|\rho(\cdot, s)\|_{L_2(\Omega_R)}^2) ds \\ &\quad + \int_0^t (\|\nabla^2 \mathbf{u}(\cdot, s)\|^2 + \|\mathbf{u}_s(\cdot, s)\|^2 + \|\nabla \rho(\cdot, s)\|^2 + \|\nabla \rho_s(\cdot, s)\|^2) ds \\ &\quad + \int_0^t (\|\nabla^3 \mathbf{u}(\cdot, s)\|^2 + \|\nabla \mathbf{u}_s(\cdot, s)\|^2 + \|\nabla^2 \rho(\cdot, s)\|^2) ds \\ &\leq C \|(\rho_0, \mathbf{u}_0)\|_{H^2(\Omega)}^2 \end{aligned} \quad (1.3)$$

for any $t > 0$ with some constant C independent of δ .

Moreover, we proved the optimal decay property.

Theorem 1.2. *Let Ω be a 3-dimensional exterior domain whose boundary Γ is a C^3 compact hyper-surface. Then, there exists a small positive number δ such that if initial data $(\rho_0, \mathbf{u}_0) \in L_1(\Omega)^4 \cap H^2(\Omega)^4$ satisfy the smallness assumption: $\|(\rho_0, \mathbf{u}_0)\|_{L_1(\Omega)} + \|(\rho_0, \mathbf{u}_0)\|_{H^2(\Omega)} \leq \delta$ as well as the compatibility condition, then problem (1.1) with $T = \infty$ admits unique solutions ρ and \mathbf{u} satisfying the same regularity conditions (1.2) and (1.3)*

in Theorem 1.1 and possessing the decay estimate:

$$\begin{aligned} \sup_{0 < s < t} (1+s)^{3/4} \|(\rho, \mathbf{u})(\cdot, s)\| + \sup_{0 < s < t} (1+s)^{5/4} \|(\nabla \rho, \nabla \mathbf{u})(\cdot, s)\| \\ + \sup_{0 < s < t} (1+s)^{5/4} \|(\nabla^2 \rho, \nabla^2 \mathbf{u})(\cdot, s)\| \\ \leq C(\|(\rho_0, \mathbf{u}_0)\|_{L_1(\Omega)} + \|(\rho_0, \mathbf{u}_0)\|_{H^2(\Omega)}) \end{aligned}$$

for any $t > 0$ with some constant C independent of δ .

2 Outline of proof of Theorem 1.1

In this section we consider the following equation:

$$\begin{cases} \rho_t + \mathbf{u} \cdot \nabla \rho + \rho_* \operatorname{div} \mathbf{u} = f_n & \text{in } \Omega \times (0, T), \\ \mathbf{u}_t - \mu_* \Delta \mathbf{u} - \nu_* \nabla \operatorname{div} \mathbf{u} + \gamma_* \nabla \rho = \mathbf{g}_n & \text{in } \Omega \times (0, T), \\ \mathbf{u}|_\Gamma = 0, \quad (\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0) & \text{in } \Omega, \end{cases} \quad (2.1)$$

where

$$\begin{aligned} \mu_* &= \mu/\rho_*, \quad \nu_* = \nu/\rho_*, \quad \gamma_* = P'(\rho_*)/\rho_*, \\ f_n &= \rho \operatorname{div} \mathbf{u}, \\ \mathbf{g}_n &= \left(\frac{1}{\rho + \rho_*} - \frac{1}{\rho_*} \right) (\mu \Delta \mathbf{u} + \nu \nabla \operatorname{div} \mathbf{u}) - \left(\frac{P'(\rho + \rho_*)}{\rho + \rho_*} - \frac{P'(\rho_*)}{\rho_*} \right) \nabla \rho. \end{aligned}$$

For the sake of simplicity, we use the abbreviation: $\|\cdot\|_{L_2(\Omega)} = \|\cdot\|$, $\|\cdot\|_{L_q(\Omega)} = \|\cdot\|_q$ ($q \neq 2$) and $\|\cdot\|_{H^s(\Omega)} = \|\cdot\|_{H^s}$ ($s = 1, 2$). We set

$$\begin{aligned} \mathbf{I}_1(t) &= \sup_{0 < s < t} \|(\rho, \mathbf{u})(\cdot, s)\|^2 + \int_0^t (\|\nabla \mathbf{u}(\cdot, s)\|^2 + \|\rho_s(\cdot, s)\|^2 + \|\rho(\cdot, s)\|_{L_2(\Omega_R)}^2) ds, \\ \mathbf{I}_2(t) &= \sup_{0 < s < t} \|(\nabla \rho, \rho_s, \nabla \mathbf{u})(\cdot, s)\|^2 \\ &\quad + \int_0^t (\|\nabla^2 \mathbf{u}(\cdot, s)\|^2 + \|\mathbf{u}_s(\cdot, s)\|^2 + \|\nabla \rho(\cdot, s)\|^2 + \|\nabla \rho_s(\cdot, s)\|^2) ds, \\ \mathbf{I}_3(t) &= \sup_{0 < s < t} \|(\nabla^2 \rho, \nabla \rho_s, \nabla^2 \mathbf{u})(\cdot, s)\|^2 \\ &\quad + \int_0^t (\|\nabla^3 \mathbf{u}(\cdot, s)\|^2 + \|\nabla \mathbf{u}_s(\cdot, s)\|^2 + \|\nabla^2 \rho(\cdot, s)\|^2) ds, \\ \mathbf{I}(t) &= \mathbf{I}_1(t) + \mathbf{I}_2(t) + \mathbf{I}_3(t). \end{aligned}$$

To prove Theorem 1.1, it suffices to prove that

$$\mathbf{I}(t) \leq K_1(\|(\rho_0, \mathbf{u}_0)\|_{H^2}^2 + \mathbf{I}(t)^{3/2} + \mathbf{I}(t)^2) \quad (2.2)$$

provided that $\|(\rho_0, \mathbf{u}_0)\|_{H^2} \leq 1$.

Multiplying the first equation in (2.1) by $\rho_*^{-1}\rho$ and the second one in (2.1) by $\gamma_*^{-1}\mathbf{u}$, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \{ \rho_*^{-1} \|\rho(\cdot, t)\|^2 + \gamma_*^{-1} \|\mathbf{u}(\cdot, t)\|^2 \} + \mu_* \|\nabla \mathbf{u}(\cdot, t)\|^2 + \nu_* \|\operatorname{div} \mathbf{u}(\cdot, t)\|^2 \\ &= \frac{1}{\rho_*} (f_n, \rho) + \frac{1}{\gamma_*} (\mathbf{g}_n, \mathbf{u}) + ((\operatorname{div} \mathbf{u})\rho, \rho). \end{aligned} \quad (2.3)$$

In what follows, to estimate the non-linear term we use the following estimates:

$$\begin{aligned} \|u\|_6 &\leq C \|\nabla u\|, & \|u\|_3 &\leq C \|u\|_{H^1}, \\ \|u\|_{L_2(\Omega_R)} &\leq C_R \|\nabla u\|, & \|u\|_\infty &\leq C \|\nabla u\|_{H^1}. \end{aligned} \quad (2.4)$$

Integrating (2.3) and using (2.4), we have

$$\|(\rho, \mathbf{u})(\cdot, t)\|^2 + \int_0^t \|\nabla \mathbf{u}(\cdot, s)\|^2 ds \leq C(\|(\rho_0, \mathbf{u}_0)\|^2 + \mathbf{I}(t)^{3/2}). \quad (2.5)$$

By the first equation in (2.1) and (2.4), we have

$$\|\rho_t\| \leq C(1 + \|\nabla \rho\|_{H^1}) \|\nabla \mathbf{u}\|, \quad (2.6)$$

so that

$$\int_0^t \|\rho_s(\cdot, s)\|^2 ds \leq C \left(\int_0^t \|\nabla \mathbf{u}(\cdot, s)\|^2 ds + \mathbf{I}(t)^2 \right). \quad (2.7)$$

To estimate $\|\rho\|_{L_2(\Omega_R)}$, we use the following lemma:

Lemma 2.1. *Let ρ and \mathbf{u} be solutions to problem (1.1) with*

$$\begin{aligned} \rho &\in C^1([0, T], H^1(\Omega)) \cap C^0([0, T], H^2(\Omega)), \\ \mathbf{u} &\in C^1([0, T], L_2(\Omega)^N) \cap C^0([0, T], H^2(\Omega)^N). \end{aligned}$$

Then, we have

$$\begin{aligned} \int_0^t \|\rho(\cdot, s)\|_{L_2(\Omega_R)}^2 ds &\leq C \int_0^t \|\nabla \rho\| \{ (1 + \|\nabla \rho\|_{H^1}) \|\nabla \mathbf{u}\| \\ &\quad + \|\mathbf{u}_s\| + \|\nabla \mathbf{u}\| \|\nabla \mathbf{u}\|_{H^1} + \|\nabla \mathbf{u}\|^{1/2} \|\nabla^2 \mathbf{u}\|^{1/2} \} ds. \end{aligned} \quad (2.8)$$

By (2.8), we see that for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ depending on ε such that

$$\begin{aligned} \int_0^t \|\rho(\cdot, s)\|_{L_2(\Omega_R)}^2 ds &\leq \varepsilon \int_0^t (\|\nabla \rho(\cdot, s)\|^2 + \|\nabla^2 \mathbf{u}(\cdot, s)\|^2) ds \\ &\quad + C_\varepsilon \int_0^t (\|\nabla \mathbf{u}(\cdot, s)\|^2 + \|\mathbf{u}_s(\cdot, s)\|^2) ds \\ &\quad + C_\varepsilon \left(\sup_{0, s < t} \|\nabla \mathbf{u}(\cdot, s)\| \right)^2 \int_0^t \|\nabla \mathbf{u}(\cdot, s)\|^2 ds. \end{aligned} \quad (2.9)$$

Next, we estimate $\|\mathbf{u}_t(\cdot, t)\|^2$. Multiplying the second equation in (2.1) by \mathbf{u}_t , we have

$$\|\mathbf{u}_t(\cdot, t)\|^2 + \frac{1}{2} \frac{d}{dt} \{\mu_* \|\nabla \mathbf{u}(\cdot, t)\|^2 + \nu_* \|\operatorname{div} \mathbf{u}(\cdot, t)\|^2\} + \gamma_* (\nabla \rho, \mathbf{u}_t) = (\mathbf{g}_n, \mathbf{u}_t). \quad (2.10)$$

By the first equation in (2.1), we have

$$(\nabla \rho, \mathbf{u}_t) = -\frac{d}{dt}(\rho, \operatorname{div} \mathbf{u}) + (\rho_t, \operatorname{div} \mathbf{u}),$$

which, combined with (2.10), furnishes that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \{\mu_* \|\nabla \mathbf{u}(\cdot, t)\|^2 + \nu_* \|\operatorname{div} \mathbf{u}(\cdot, t)\|^2 - 2(\rho, \operatorname{div} \mathbf{u})\} + \frac{1}{2} \|\mathbf{u}_t(\cdot, t)\|^2 \\ & \leq \frac{1}{2} (\|\mathbf{g}_n(\cdot, t)\|^2 + \|\rho_t(\cdot, t)\|^2 + \|\nabla \mathbf{u}(\cdot, t)\|^2). \end{aligned} \quad (2.11)$$

By (2.4) and (2.6), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \{\mu_* \|\nabla \mathbf{u}(\cdot, t)\|^2 + \nu_* \|\operatorname{div} \mathbf{u}(\cdot, t)\|^2 - 2(\rho, \operatorname{div} \mathbf{u})\} + \frac{1}{2} \|\mathbf{u}_t(\cdot, t)\|^2 \\ & \leq C \{\|\nabla \rho\|_{H^1}^2 (\|\nabla \mathbf{u}\|_{H^1}^2 + \|\nabla \rho\|^2) + \|\nabla \mathbf{u}\|^2\}. \end{aligned} \quad (2.12)$$

Integrating (2.12), we have

$$\begin{aligned} \|\nabla \mathbf{u}(\cdot, t)\|^2 + \int_0^t \|\mathbf{u}_s(\cdot, s)\|^2 ds & \leq C \{\|\nabla \mathbf{u}_0\|^2 + \|\rho_0\|^2 + \|\rho(\cdot, t)\|^2 + \int_0^t \|\nabla \mathbf{u}(\cdot, s)\|^2 ds \\ & \quad + (\sup_{0 < s < t} \|\nabla \rho(\cdot, s)\|_{H^1})^2 \int_0^t (\|\nabla \mathbf{u}(\cdot, s)\|_{H^1}^2 + \|\nabla \rho(\cdot, s)\|^2) ds\}. \end{aligned} \quad (2.13)$$

By (2.7), (2.9) and (2.13), we have

$$\begin{aligned} & \int_0^t (\|\rho_s(\cdot, s)\|^2 + \|\rho(\cdot, s)\|_{L^2(\Omega_R)}^2) ds \leq \varepsilon \int_0^t (\|\nabla^2 \mathbf{u}(\cdot, s)\|^2 + \|\nabla \rho(\cdot, s)\|^2) ds \\ & \quad + C_\varepsilon (\|(\nabla \mathbf{u}_0, \rho_0)\|^2 + \|\rho(\cdot, t)\|^2 + \int_0^t \|\nabla \mathbf{u}(\cdot, s)\|^2 ds + \mathbf{I}(t)^2), \end{aligned}$$

which, combined with (2.5), furnishes that

$$\mathbf{I}_1(t) \leq C_\varepsilon (\|(\rho_0, \mathbf{u}_0)\|_{H^1}^2 + \mathbf{I}(t)^{3/2} + \mathbf{I}(t)^2) + \varepsilon \mathbf{I}_2(t). \quad (2.14)$$

We estimate ρ_t , \mathbf{u}_t and $\nabla \mathbf{u}_t$. Differentiating the equations in (2.1) once with respect to t , we have

$$\begin{cases} \partial_t(\rho_t) + \mathbf{u} \cdot \nabla(\rho_t) + \rho_* \operatorname{div} \mathbf{u}_t = \partial_t f_n - \mathbf{u}_t \cdot \nabla \rho & \text{in } \Omega \times (0, T), \\ \partial_t(\mathbf{u}_t) - \mu_* \Delta \mathbf{u}_t - \nu_* \nabla \operatorname{div} \mathbf{u}_t + \gamma_* \nabla \rho_t = \partial_t \mathbf{g}_n & \text{in } \Omega \times (0, T), \\ \mathbf{u}_t|_\Gamma = 0, \quad (\rho_t, \mathbf{u}_t)|_{t=0} = (\rho_1, \mathbf{u}_1) & \text{in } \Omega, \end{cases}$$

where we have set

$$\begin{aligned}\rho_1 &= -\operatorname{div}((\rho_* + \rho_0)\mathbf{u}_0), \\ \mathbf{u}_1 &= -\mathbf{u}_0 \cdot \nabla \mathbf{u}_0 + \frac{1}{\rho_* + \rho_0}(\mu \Delta \mathbf{u}_0 + \nu \nabla \operatorname{div} \mathbf{u}_0 - P'(\rho_* + \rho) \nabla \rho).\end{aligned}$$

Analogously to (2.3), we have

$$\begin{aligned}\frac{1}{2} \frac{d}{dt} \{ \rho_*^{-1} \|\rho_t(\cdot, t)\|^2 + \gamma_*^{-1} \|\mathbf{u}_t(\cdot, t)\|^2 \} &+ \mu_* \|\nabla \mathbf{u}_t(\cdot, t)\|^2 + \nu_* \|\operatorname{div} \mathbf{u}_t(\cdot, t)\|^2 \\ &= \frac{1}{\rho_*} (\partial_t f_n - \mathbf{u}_t \cdot \nabla \rho, \rho_t) + \frac{1}{\gamma_*} (\partial_t \mathbf{g}_n, \mathbf{u}_t) + ((\operatorname{div} \mathbf{u}) \rho_t, \rho_t).\end{aligned}\quad (2.15)$$

By (2.4),

$$\begin{aligned}\frac{1}{2} \frac{d}{dt} \{ \rho_*^{-1} \|\rho_t(\cdot, t)\|^2 + \gamma_*^{-1} \|\mathbf{u}_t(\cdot, t)\|^2 \} &+ \mu_* \|\nabla \mathbf{u}_t(\cdot, t)\|^2 + \nu_* \|\operatorname{div} \mathbf{u}_t(\cdot, t)\|^2 \\ &\leq C(\|\nabla \mathbf{u}_t\|^2 \|\nabla \rho\|_{H^1} + \|\nabla \mathbf{u}_t\| \|\nabla \rho\| \|\rho_t\|_{H^1} \\ &\quad + \|\mathbf{u}_t\| \|\nabla \rho\| \|\nabla \rho_t\| + \|\nabla \mathbf{u}\| \|\rho_t\|_{H^1}^2).\end{aligned}\quad (2.16)$$

Integrating (2.16), we have

$$\begin{aligned}\|(\rho_t, \mathbf{u}_t)(\cdot, t)\|^2 &+ \int_0^t \|\nabla \mathbf{u}_s(\cdot, s)\|^2 ds \\ &\leq C\{ \|(\rho_1, \mathbf{u}_1)\|^2 + (\sup_{0 < s < t} \|\nabla \rho(\cdot, s)\|_{H^1}) \int_0^t (\|\mathbf{u}_s(\cdot, s)\|_{H^1}^2 + \|\nabla \rho_s(\cdot, s)\|^2) ds \\ &\quad + (\sup_{0 < s < t} \|\nabla \mathbf{u}(\cdot, s)\|) \int_0^t \|\rho_s(\cdot, s)\|_{H^1}^2 ds \}.\end{aligned}\quad (2.17)$$

To prove the estimate of the higher order derivatives, we localize the problem in the whole space and near the boundary. Then, we consider the whole space problem:

$$\begin{cases} \rho_t + \mathbf{u} \cdot \nabla \rho + \rho_* \operatorname{div} \mathbf{v} = f & \text{in } \mathbb{R}^3, \\ \mathbf{v}_t - \mu_* \Delta \mathbf{v} - \nu_* \nabla \operatorname{div} \mathbf{v} + \gamma_* \nabla \rho = \mathbf{g} & \text{in } \mathbb{R}^3 \end{cases}\quad (2.18)$$

and the half space problem:

$$\begin{cases} \rho_t + \mathbf{u} \cdot \nabla \rho + \rho_* \operatorname{div} \mathbf{v} = f & \text{in } \mathbb{R}_+^3, \\ \mathbf{v}_t - \mu_* \Delta \mathbf{v} - \nu_* \nabla \operatorname{div} \mathbf{v} + \gamma_* \nabla \rho = \mathbf{g} & \text{in } \mathbb{R}_+^3, \\ \mathbf{v}|_{x_3=0} = 0 \end{cases}\quad (2.19)$$

where $\mathbb{R}_+^3 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$.

In the half space problem, the points are estimations of $\partial_3 \rho$ and $\partial_3^2 \rho$. To do this, we use the idea due to Matsumura-Nishida [7]. From the third component in the second equation in (2.19) we have

$$\partial_3^2 v_3 = \frac{1}{\mu_* + \nu_*} \left\{ (v_3)_t - \mu_* \sum_{i=1}^2 \partial_i^2 v_3 - \nu_* \sum_{i=1}^2 \partial_i \partial_3 v_i - g_3 \right\} + \frac{\gamma_*}{\mu_* + \nu_*} \partial_3 \rho. \quad (2.20)$$

Differentiating the first equation in (2.19) with respect to x_3 and inserting the formula (2.20), we have

$$\begin{aligned} & \partial_3(\rho_t + \mathbf{u} \cdot \nabla \rho) + \delta_* \partial_3 \rho \\ &= \partial_3 f - \frac{\rho_* \nu_*}{\mu_* + \nu_*} \sum_{i=1}^2 \partial_i \partial_3 v_i + \frac{\rho_* \mu_*}{\mu_* + \nu_*} \sum_{i=1}^2 \partial_i^2 v_3 - \frac{\rho_*}{\mu_* + \nu_*} (v_3)_t + \frac{\rho_*}{\mu_* + \nu_*} g_3 \end{aligned} \quad (2.21)$$

where $\delta_* = \rho_* \gamma_* / (\mu_* + \nu_*)$. Multiplying (2.21) by $\partial_3 \rho$, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_3 \rho(\cdot, t)\|_{L_2(\mathbb{R}_+^3)}^2 + \frac{\delta_*}{2} \|\partial_3 \rho(\cdot, t)\|_{L_2(\mathbb{R}_+^3)}^2 \\ & \leq C \left(\|\nabla f\|_{L_2(\mathbb{R}_+^3)}^2 + \|\mathbf{g}\|_{L_2(\mathbb{R}_+^3)}^2 + \|\nabla \nabla' \mathbf{v}\|_{L_2(\mathbb{R}_+^3)}^2 + \|\mathbf{v}_t\|_{L_2(\mathbb{R}_+^3)}^2 + \|\nabla \mathbf{u} \cdot \nabla \rho\|_{L_2(\mathbb{R}_+^3)}^2 \right) \end{aligned}$$

where $\nabla \nabla' w = (\partial_i \partial_j w \mid i = 1, 2, 3, j = 1, 2)$. Differentiating (2.21) with respect to x_3 , we have

$$\begin{aligned} & \partial_3^2(\rho_t + \mathbf{u} \cdot \nabla \rho) + \delta_* \partial_3^2 \rho \\ &= \partial_3^2 f - \partial_3 \left(\frac{\rho_* \nu_*}{\mu_* + \nu_*} \sum_{i=1}^2 \partial_i \partial_3 v_i + \frac{\rho_* \mu_*}{\mu_* + \nu_*} \sum_{i=1}^2 \partial_i^2 v_3 - \frac{\rho_*}{\mu_* + \nu_*} (v_3)_t + \frac{\rho_*}{\mu_* + \nu_*} g_3 \right) \end{aligned} \quad (2.22)$$

and therefore,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_3^2 \rho(\cdot, t)\|_{L_2(\mathbb{R}_+^3)}^2 + \delta_* \|\partial_3^2 \rho(\cdot, t)\|_{L_2(\mathbb{R}_+^3)}^2 \\ & \leq C \left(\|(\operatorname{div} \mathbf{u}) \cdot \partial_3^2 \rho\|_{L_2(\mathbb{R}_+^3)}^2 + \|\nabla \mathbf{u} \cdot \nabla^2 \rho\|_{L_2(\mathbb{R}_+^3)}^2 + \|\nabla^2 \mathbf{u} \cdot \nabla \rho\|_{L_2(\mathbb{R}_+^3)}^2 \right. \\ & \quad \left. + \|\nabla^2 f\|_{L_2(\mathbb{R}_+^3)}^2 + \|\nabla^2 \nabla' \mathbf{v}\|_{L_2(\mathbb{R}_+^3)}^2 + \|\nabla \mathbf{v}_t\|_{L_2(\mathbb{R}_+^3)}^2 + \|\nabla \mathbf{g}\|_{L_2(\mathbb{R}_+^3)}^2 \right) \end{aligned}$$

where $\nabla^2 \nabla' \mathbf{v} = (\nabla^2 \partial_1 \mathbf{v}, \nabla^2 \partial_2 \mathbf{v})$.

Using Sobolev's embedding theorem and (2.4), we have

$$\begin{aligned} & \int_0^t \|\nabla^2 \rho(\cdot, s) \nabla \mathbf{u}(\cdot, s)\|^2 ds \leq \int_0^t \|\nabla^2 \rho(\cdot, s)\|^2 \|\nabla \mathbf{u}(\cdot, s)\|_\infty^2 ds \\ & \leq C \int_0^t \|\nabla^2 \rho(\cdot, s)\|^2 \|\nabla \mathbf{u}(\cdot, s)\|_{H^2}^2 ds \\ & \leq C \sup_{0 < s < t} \|\nabla^2 \rho(\cdot, s)\|^2 \int_0^t \|\nabla \mathbf{u}(\cdot, s)\|_{H^2}^2 ds. \end{aligned}$$

In this way, we can enclose the estimation as the gradient \mathbf{u} belongs to $L_2((0, \infty), H^2(\Omega))$ and ρ belongs to $L_\infty((0, \infty), H^2(\Omega))$.

Applying the same argument as in the above and using the cut-off technique, we have

$$\begin{aligned} & \|(\nabla \rho, \nabla \mathbf{u})(\cdot, t)\|^2 + \int_0^t \|(\nabla^2 \mathbf{u}, \mathbf{u}_s, \nabla \rho, \nabla \rho_s)(\cdot, s)\|^2 ds \\ & \leq C \{ \|(\rho_0, \mathbf{u}_0)\|_{H^1}^2 + \mathbf{I}_1(t) + \mathbf{I}(t)^2 \}, \end{aligned}$$

which, combined with (2.17), furnishes that

$$\mathbf{I}_2(t) \leq C\{\|(\rho_0, \mathbf{u}_0)\|_{H^1}^2 + \mathbf{I}_1(t) + \mathbf{I}(t)^2\}. \quad (2.23)$$

Therefore, choosing $\varepsilon > 0$ small enough in (2.14), by (2.23) we have

$$\mathbf{I}_1(t) + \mathbf{I}_2(t) \leq C\{\|(\rho_0, \mathbf{u}_0)\|_{H^1}^2 + \mathbf{I}(t)^{3/2} + \mathbf{I}(t)^2\}. \quad (2.24)$$

Analogously to (2.24), we have

$$\mathbf{I}_3(t) \leq C\{\|(\rho_0, \mathbf{u}_0)\|_{H^2}^2 + \mathbf{I}_1(t) + \mathbf{I}_2(t) + \mathbf{I}(t)^2\} \quad (2.25)$$

provided that $\|(\rho_0, \mathbf{u}_0)\|_{H^2} \leq 1$. Combining (2.24) and (2.25), we have (2.2). This completes the proof of Theorem 1.1.

3 Outline of proof of Theorem 1.2

We set

$$\begin{aligned} \mathbf{D}_0(t) &= \sup_{0 < s < t} (1+s)^{3/4} \|(\rho, \mathbf{u})(\cdot, s)\|, \\ \mathbf{D}_1(t) &= \sup_{0 < s < t} (1+s)^{5/4} \|(\nabla \rho, \nabla \mathbf{u})(\cdot, s)\|, \\ \mathbf{D}_2(t) &= \sup_{0 < s < t} (1+s)^{5/4} \|(\nabla^2 \rho, \nabla^2 \mathbf{u})(\cdot, s)\|, \\ \mathbf{D}(t) &= \mathbf{D}_0(t) + \mathbf{D}_1(t) + \mathbf{D}_2(t). \end{aligned}$$

To prove Theorem 1.2, it suffices to prove that

$$\mathbf{D}(t) \leq K_2(\|(\rho_0, \mathbf{u}_0)\|_1 + \|(\rho_0, \mathbf{u}_0)\|_{H^2} + \mathbf{D}(t)^2) \quad (3.1)$$

with some constant K_2 with the help of L_p - L_q decay estimate for the linearized problem.

First of all, we introduce L_p - L_q decay estimate. To do this, we consider the following linearized problem:

$$\begin{cases} \rho_t + \gamma \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega \times (0, \infty), \\ \mathbf{v}_t - \alpha \Delta \mathbf{v} - \beta \nabla \operatorname{div} \mathbf{v} + \gamma \nabla \rho = 0 & \text{in } \Omega \times (0, \infty), \\ \mathbf{v}|_{\Gamma} = 0, \quad (\rho, \mathbf{v})|_{t=0} = (\rho_0, \mathbf{v}_0) & \text{in } \Omega, \end{cases} \quad (3.2)$$

where α , β and γ are positive constants. Let m be a non-negative integer. We assume that Γ is a compact $C^{m+1,1}$ hyper-surface. Let A be an operator defined by the formula:

$$A(\rho, \mathbf{v}) = (\gamma \operatorname{div} \mathbf{v}, -\alpha \Delta \mathbf{v} - \beta \nabla \operatorname{div} \mathbf{v} + \gamma \nabla \rho)$$

for any element (ρ, \mathbf{v}) of the domain $W_{p,0}^{m+1,m+2}(\Omega) = \{(\rho, \mathbf{v}) \in W_p^{m+1}(\Omega) \times W_p^{m+2}(\Omega) \mid \mathbf{v}|_{\Gamma} = 0\}$. By Shibata and Tanaka [9] we know the generation of C_0 semigroup $\{T(t)\}_{t \geq 0}$ on $W_p^{m+1,m}(\Omega) = W_p^{m+1}(\Omega) \times W_p^m(\Omega)$, which is analytic. For the solution ρ and \mathbf{v} of the equation (3.2), the following L_p - L_q decay estimate holds.

Theorem 3.1. *Let Ω be a 3-dimensional exterior domain whose boundary Γ is a C^3 compact hyper-surface. Let p and q be indices such that $1 \leq q \leq 2 \leq p < \infty$. Let*

$$[(f, \mathbf{g})]_{p,q} = \|(f, \mathbf{g})\|_{L_q(\Omega)} + \|(f, \mathbf{g})\|_{W_p^{1,0}(\Omega)}.$$

Then, for any $(f, \mathbf{g}) \in W_p^{1,0}(\Omega) \cap L_q(\Omega)^4$ and $t \geq 1$ we have

$$\begin{aligned} \|T(t)(f, \mathbf{g})\|_{L_p(\Omega)} &\leq Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}[(f, \mathbf{g})]_{p,q}, \\ \|\nabla T(t)(f, \mathbf{g})\|_{L_p(\Omega)} &\leq C \begin{cases} t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}[(f, \mathbf{g})]_{p,q} & p \leq 3, \\ t^{-\frac{3}{2q}}[(f, \mathbf{g})]_{p,q} & p \geq 3, \end{cases} \\ \|\nabla^2 P_{\mathbf{v}} T(t)(f, \mathbf{g})\|_{L_p(\Omega)} &\leq Ct^{-\frac{3}{2q}}[(f, \mathbf{g})]_{p,q} \end{aligned}$$

where $P_{\mathbf{v}}$ is the projection acting on (ρ, \mathbf{v}) defined by $P_{\mathbf{v}}(\rho, \mathbf{v}) = \mathbf{v}$.

We go back to the proof of the inequality (3.1). First, we estimate $\mathbf{D}_0(t)$ and $\mathbf{D}_1(t)$ with the help of L_p - L_q decay estimate for the linearized problem. Let $\{T(t)\}_{t \geq 0}$ be the analytic semigroup associated with the linearized problem:

$$\begin{cases} \rho_t + \rho_* \operatorname{div} \mathbf{u} = f_n - \mathbf{u} \cdot \nabla \rho & \text{in } \Omega \times (0, T), \\ \mathbf{u}_t - \mu_* \Delta \mathbf{u} - \nu_* \nabla \operatorname{div} \mathbf{u} + \gamma_* \nabla \rho = \mathbf{g}_n & \text{in } \Omega \times (0, T), \\ \mathbf{u}|_{\Gamma} = 0, \quad (\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0) & \text{in } \Omega. \end{cases} \quad (3.3)$$

Then, we have $(\rho, \mathbf{u})(\cdot, t) = T(t)(\rho_0, \mathbf{u}_0) + \mathbf{U}(t)$ with

$$\mathbf{U}(t) = \int_0^t T(t-s)((f_n - \mathbf{u} \cdot \nabla \rho)(\cdot, s), \mathbf{g}_n(\cdot, s)) ds.$$

Here, we write $H^{1,0} = W_2^{1,0}(\Omega)$ and $\|\cdot\|_{H^s} = \|\cdot\|_{H^s(\Omega)}$ for $s = 1, 2$. By Theorem 3.1, we have

$$\begin{aligned} \|T(t)(\rho_0, \mathbf{u}_0)\| &\leq C(1+t)^{-\frac{3}{4}}(\|(\rho_0, \mathbf{u}_0)\|_1 + \|(\rho_0, \mathbf{u}_0)\|_{H^{1,0}}), \\ \|\nabla T(t)(\rho_0, \mathbf{u}_0)\| &\leq C(1+t)^{-\frac{5}{4}}(\|(\rho_0, \mathbf{u}_0)\|_1 + \|(\rho_0, \mathbf{u}_0)\|_{H^1}) \end{aligned} \quad (3.4)$$

for any $t > 0$. To estimate $\mathbf{U}(t)$, we observe that

$$\begin{aligned} \|(f_n - \mathbf{u} \cdot \nabla \rho)(\cdot, s)\|_1 + \|(f_n - \mathbf{u} \cdot \nabla \rho)(\cdot, s)\|_{H^1} &\leq C(1+s)^{-2} \mathbf{D}(s)^2, \\ \|\mathbf{g}_n(s)\|_1 + \|\mathbf{g}_n(s)\| &\leq C(1+s)^{-2} \mathbf{D}(s)^2. \end{aligned}$$

Thus, applying the L_p - L_q decay estimate and the usual analytic semi-group estimate, we have

$$\begin{aligned} \|\mathbf{U}(t)\| &\leq C \left\{ \int_0^{t-1} (t-s)^{-\frac{3}{4}}(1+s)^{-2} ds + \int_{t-1}^t (1+s)^{-2} ds \right\} \mathbf{D}(t)^2 \\ &\leq Ct^{-\frac{3}{4}} \mathbf{D}(t)^2, \\ \|\nabla \mathbf{U}(t)\| &\leq C \left\{ \int_0^{t-1} (t-s)^{-\frac{5}{4}}(1+s)^{-2} ds + \int_{t-1}^t (t-s)^{-\frac{1}{2}}(1+s)^{-2} ds \right\} \mathbf{D}(t)^2 \\ &\leq Ct^{-\frac{5}{4}} \mathbf{D}(t)^2 \end{aligned} \quad (3.5)$$

for $t \geq 1$. Since $\|(\rho, \mathbf{u})(\cdot, t)\|_{H^2} \leq C\|(\rho_0, \mathbf{u}_0)\|_{H^2}$ as follows from (1.3), by (3.4) and (3.5)

$$\mathbf{D}_0(t) + \mathbf{D}_1(t) \leq C\{\|(\rho_0, \mathbf{u}_0)\|_1 + \|(\rho_0, \mathbf{u}_0)\|_{H^2} + \mathbf{D}(t)^2\}. \quad (3.6)$$

Next, we estimate $\|(\rho_t, \mathbf{u}_t)(\cdot, t)\|$. For this purpose, we use

Lemma 3.2. *Let $f(t)$ be a non-negative $C^1([0, \infty))$ function and let $g_i(t)$ ($i = 1, 2, 3, 4$) be non-negative functions such that $g_i \in C^0((0, \infty))$ ($i = 1, 2, 3$) and $g_4 \in L_2((0, \infty))$. Assume that*

$$\frac{d}{dt}f(t) + cf(t) \leq g_1(t) + g_2(t) + g_3(t)g_4(t)$$

for any $t > T_0$ with some constant $c > 0$. Then, for any $\alpha > 0$ there exists a $T_1 \geq T_0$ such that

$$\begin{aligned} (1+t)^\alpha f(t) &\leq (1+T_1)^\alpha f(T_1) + (2/c) \left(\sup_{T_1 < s < t} (1+s)^\alpha g_1(s) \right) \\ &\quad + \int_{T_1}^t (1+s)^\alpha g_2(s) ds + \sqrt{1/c} \left(\sup_{T_1 < s < t} (1+s)^\alpha g_3(s) \right) \left(\int_{T_1}^t g_4(s)^2 ds \right)^{1/2}. \end{aligned}$$

In view of Theorem 1.1, we may assume that

$$\begin{aligned} \int_0^t (\|\nabla \mathbf{u}(\cdot, s)\|_{H^2}^2 + \|\nabla \rho(\cdot, s)\|_{H^1}^2 + \|\rho_s(\cdot, s)\|_{H^1}^2 + \|\mathbf{u}_s(\cdot, s)\|_{H^1}^2) ds \\ + \|\rho_t(\cdot, t)\|_{H^1}^2 + \|(\rho, \mathbf{u})(\cdot, t)\|_{H^2}^2 \leq C\|(\rho_0, \mathbf{u}_0)\|_{H^2}^2 \leq \varepsilon \end{aligned} \quad (3.7)$$

for any $t > 0$ with some small $\varepsilon > 0$ which is decided later. We choose $\varepsilon > 0$ small enough eventually, so that we may assume that $0 < \varepsilon < 1$. By (2.6) and (3.7), we have

$$\|\rho_t(\cdot, t)\| \leq C\|\nabla \mathbf{u}(\cdot, t)\|. \quad (3.8)$$

Let κ be a small positive number $> \varepsilon$ determined later. By (2.11) and (2.15),

$$\begin{aligned} \frac{d}{dt} \{ &\mu_* \|\nabla \mathbf{u}(\cdot, t)\|^2 + \nu_* \|\operatorname{div} \mathbf{u}(\cdot, t)\|^2 + \rho_*^{-1} \|\rho_t(\cdot, t)\|^2 + \gamma_*^{-1} \|\mathbf{u}_t(\cdot, t)\|^2 \} \\ &+ \kappa \{ \mu_* \|\nabla \mathbf{u}(\cdot, t)\|^2 + \nu_* \|\operatorname{div} \mathbf{u}(\cdot, t)\|^2 + \rho_*^{-1} \|\rho_t(\cdot, t)\|^2 + \gamma_*^{-1} \|\mathbf{u}_t(\cdot, t)\|^2 \} \\ &\leq C\kappa (\|\nabla \mathbf{u}(\cdot, t)\|^2 + \|\rho_t(\cdot, t)\|^2 + \|\mathbf{u}_t(\cdot, t)\|^2) - \min\left(\frac{1}{2}, \mu_*\right) \|\mathbf{u}_t(\cdot, t)\|_{H^1}^2 \\ &+ C(\|\mathbf{g}_n(\cdot, t)\|^2 + \|\rho_t(\cdot, t)\|^2 + \|\nabla \mathbf{u}(\cdot, t)\|^2 + |(\partial_t f_n, \rho_t)| \\ &+ |(\mathbf{u}_t \cdot \nabla \rho, \rho_t)| + |(\partial_t \mathbf{g}_n, \mathbf{u}_t)| + |((\operatorname{div} \mathbf{u})\rho_t, \rho_t)|) \end{aligned} \quad (3.9)$$

Combining (3.7), (3.8), (3.9) and (2.4), choosing $\kappa > 0$ in such a way that $C\kappa < \min(1/2, \mu_*)$ with some constant independent of κ , and setting

$$\begin{aligned} f(t) &= \{\mu_* \|\nabla \mathbf{u}(\cdot, t)\|^2 + \nu_* \|\operatorname{div} \mathbf{u}(\cdot, t)\|^2 + \rho_*^{-1} \|\rho_t(\cdot, t)\|^2 + \gamma_*^{-1} \|\mathbf{u}_t(\cdot, t)\|^2\}, \\ g_1(t) &= C\kappa (\|\nabla \mathbf{u}(\cdot, t)\|^2 + \|\nabla \rho(\cdot, t)\|^2), \\ g_2(t) &= C\kappa \{(\|\nabla \mathbf{u}(\cdot, t)\|_{H^2}^2 + \|\nabla \rho(\cdot, t)\|_{H^1}^2) \|\nabla \rho(\cdot, t)\|^2 + \|\nabla \mathbf{u}(\cdot, t)\|_{H^2}^2 \|\rho_t(\cdot, t)\|^2\}, \end{aligned}$$

we have

$$\frac{d}{dt}f(t) + \kappa f(t) \leq g_1(t) + g_2(t).$$

Thus, applying Lemma 3.2 and using Theorem 1.1 to estimate the term corresponding to $(1 + T_1)^\alpha f(T_1)$, we have

$$\begin{aligned} (1+t)^{5/2} \|(\rho_t, \mathbf{u}_t)(\cdot, t)\|^2 &\leq C_\kappa \{ (\|(\rho_0, \mathbf{u}_0)\|_{H^2}^2 + \sup_{0 < s < t} (1+s)^{5/2} \|(\nabla \rho, \nabla \mathbf{u})(\cdot, s)\|^2 \\ &+ (\sup_{0 < s < t} (1+s)^{5/2} \|\nabla \rho(\cdot, s)\|^2) \int_0^t (\|\nabla \mathbf{u}(\cdot, s)\|_{H^2}^2 + \|\nabla \rho(\cdot, s)\|_{H^1}^2) ds \\ &+ (\sup_{0 < s < t} (1+s)^{5/2} \|\rho_s(\cdot, s)\|^2) \int_0^t \|\nabla \mathbf{u}(\cdot, s)\|_{H^2}^2 ds \}. \end{aligned}$$

Since

$$C_\kappa (\sup_{0 < s < t} (1+s)^{5/2} \|\rho_s(\cdot, s)\|^2) \int_0^t \|\nabla \mathbf{u}(\cdot, s)\|_{H^2}^2 ds \leq C_\kappa \varepsilon (\sup_{0 < s < t} (1+s)^{5/2} \|\rho_s(\cdot, s)\|^2)$$

as follows from (3.7), after fixing κ , we choose $\varepsilon < \kappa$ in such a way that $C_\kappa \varepsilon \leq 1/2$, we have

$$(1+t)^{5/4} \|(\rho_t, \mathbf{u}_t)(\cdot, t)\| \leq C(\|(\rho_0, \mathbf{u}_0)\| + \mathbf{D}_1(t)). \quad (3.10)$$

Applying the elliptic estimate to the second equation in (3.3) and using (3.10), we have

$$(1+t)^{5/4} \|\nabla^2 \mathbf{u}(\cdot, t)\| \leq C(\|(\rho_0, \mathbf{u}_0)\|_{H^2} + \mathbf{D}_1(t)). \quad (3.11)$$

Finally, we estimate $\|\nabla^2 \rho\|$. To do this, we consider the whole space problem (2.18) and the half space problem (2.19) and analogously to (3.11), we have

$$(1+t)^{5/4} \|\nabla^2 \rho(\cdot, t)\| \leq C(\|(\rho_0, \mathbf{u}_0)\|_{H^2} + \mathbf{D}_1(t)). \quad (3.12)$$

Thus, by (3.6), (3.10), (3.11) and (3.12), we have (3.1), which completes the proof of Theorem 1.2.

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